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Oscillation of even order nonlinear dynamic equations on time-scales

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Abstract. In this paper, the authors discuss the oscillatory behavior of solutions to a class of even order nonlinear dynamic equations on time scales. The results are established by a comparison with n-th order delay dynamic inequalities or first-order delay dynamic equations whose oscillatory characters are known. Several corollaries are obtained for special cases.

1. Introduction

In this paper we discuss the oscillatory behavior of solutions of the even-order nonlinear dynamic equation

$$\Big(b(\xi)\Big(\phi^{\Delta^{n-1}}(\xi)\Big)^{\alpha}\Big)^{\Delta}+q(\xi)\phi^{\gamma}(\tau(\xi))=0,\ \xi\in\mathbb{T}, \tag{1}$$

where $\mathbb{T}$ is an arbitrary time scale (i.e., a nonempty closed subset of the real numbers) with $\sup\mathbb{T}=+\infty$, and $n\geq 4$ is an even positive integer. We denote time scale intervals by $[\xi_0,\infty)_{\mathbb{T}}:=[\xi_0,\infty)\cap\mathbb{T}$. For basic concepts, terminology, and notation for the time scale calculus, we refer the reader to the fundamental works of Bohner and Peterson [8,9]. In general, we will use these notions without further explanation.

Throughout we will assume that the conditions below are satisfied:

(H1) α and γ are the ratios of odd positive integers with $\alpha>1$;
(H2) q, $b\in C_{rd}([\xi_0,\infty)_{\mathbb{T}},(0,\infty))$;
(H3) $\tau\in C_{rd}([\xi_0,\infty)_{\mathbb{T}},(0,\infty)_{\mathbb{T}})$ satisfies $\tau(\xi)\leq\xi$ and $\tau(\xi)\to\infty$ as $\xi\to\infty$;
(H4) $b^{\Delta}(\xi)\geq 0$ and

$$B(\xi,\xi_0)=\int_{\xi_0}^{\xi}b^{-\frac{1}{\alpha}}(s)\Delta s\to\infty \text{ as } \xi\to\infty.$$

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By a *solution* of (1), we mean by a nontrivial real-valued function ϕ that satisfies equation (1) for $\xi \geq \xi_0$. We consider only nontrivial continuable solutions, i.e., those that do not vanish in some neighborhood of infinity. An *oscillatory solution* of equation (1) is a function that is neither eventually positive nor eventually negative. The solutions which do not satisfy this condition are termed *nonoscillatory.* Equation (1) is called oscillatory if all solutions are oscillatory. Our aim here is to prove new oscillation criteria for equation (1).

Many books and numerous articles have been written on oscillatory behavior of solutions of differential equations of different types and orders, and we mention [4–7, 25, 26, 28, 29, 31] as some typical examples. In recent years, there has been significant amount of research on the oscillatory behavior of solutions of various class of dynamic equations on time scales; for example, [1, 3, 14–24] and the references therein for some recent research on this topic.

Other authors have studied equations in the form of (1) with the goal of obtaining sufficient conditions for the oscillation of all solutions. For example, Chen and Nie [11] considered an equation similar to (1) with a sum of terms, rather than a single one, on the left hand side. In [13, 19] the authors considered equation (1) with $b(t) \equiv 1$ and $\alpha = 1$. Grace *et al.* studied equation (1) and obtained integral conditions, not comparison results, that guaranteed the oscillation of all solutions. The advantage of comparison type theorems is that various existing results that guarantee the desired behavior of the comparison equation can be applied.

To the best of our knowledge, there are no reported results concerning the oscillation of the dynamic equation (1) via a comparison with inequalities of the form

$$\left(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\right)^{\Delta} + Q(\xi)\phi^{\gamma+1-\alpha}(\tau(\xi)) \leq 0, \tag{2}$$

or first order equations of type

$$\psi^{\Delta}(\xi) + P(\xi)\psi^{\gamma+1-\alpha}(\tau(\xi)) = 0, \tag{3}$$

where the functions P, $Q \in C_{rd}((\xi_0, \infty)_{\mathbb{T}}, (0, \infty))$ are appropriately chosen.

Motivated by this observation, our aims is to establish some new oscillation criteria for equation (1) via a comparison with the dynamic inequalities or dynamic equations of the types (2) and (3) whose oscillatory characters are known.

2. PRELIMINARIES

In this section we present some topics that are needed to prove our main results.

The Taylor monomials $\{h_n(\xi, s)\}_{n=0}^{\infty}$ are defined recursively as:

$$h_0(\xi, s) = 1 \quad \text{and} \quad h_{n+1}(\xi, s) = \int_s^{\xi} h_n(\tau, s)\Delta\tau, \quad \text{for} \quad \xi,\ s \in \mathbb{T}, \quad n \geq 1.$$

It is easy to see that $h_1(\xi, s) = \xi - s$ for any time scale, but in general it is not easy to find h_n for $n \geq 2$. For some particular time scales this is possible. For example, $h_n(\xi, s) = \frac{(\xi-s)^n}{n!}$ for ξ, $s \in \mathbb{R}$, and $h_n(\xi, s) = \frac{(\xi-s)^{\underline{n}}}{n!}$ for ξ, $s \in \mathbb{Z}$, where $\xi^{\underline{n}} = \xi(\xi+1)\cdots(\xi+n-1)$ is the so-called falling function. We also know that

$$0 \leq h_n(\xi, s) \leq (\xi - s)^n, \quad \text{for} \quad \xi \geq s \quad \text{and} \quad n = 0, 1, \ldots.$$

For additional details, see [8, Section 1.6].

The following lemmas play important roles in establishing our results.

Lemma 1 (Kiguradze's Theorem [1, Theorem 5])**.** *Let* $\sup \mathbb{T} = \infty$, $n \in \mathbb{N}$, *and* $\phi \in C^n_{rd}(\mathbb{T}, \mathbb{R}^+)$. *Assume that* $\phi^{\Delta^n}(t) \neq 0$ *is either nonpositive or nonnegative on* $\mathbb{T}$. *Then there is* $m \in (0, n) \cap \mathbb{Z}$ *with* $(-1)^{n-m}\phi^{\Delta^n}(\xi) \geq 0$ *for each sufficiently large* ξ. *In addition, the following conditions hold for* ξ *sufficiently large:*

(i) *For* $0 \leq k < m$, *we have* $\phi^{\Delta^k}(\xi) > 0$.
(ii) *For* $m \leq k < n$, *we have* $(-1)^{m-k}\phi^{\Delta^k}(\xi) > 0$.

The following lemma can be found in [13, Lemma 2.8], [15, Lemma 2.2], and [16, Lemma 2.3].

Lemma 2. *Let* $\sup \mathbb{T} = \infty$ *and* $\phi \in C^n_{rd}(\mathbb{T}, (0, \infty))$, $n \geq 2$. *Assume Kiguradze's Theorem holds for some* $m \in (0, n) \cap \mathbb{Z}$ *and* $\phi^{\Delta^n}(\xi) \leq 0$ *on* $\mathbb{T}$. *Then, there is a sufficiently large* $\xi_1 \in \mathbb{T}$ *such that*

$$\phi^{\Delta}(\xi) \geq h_{m-1}(\xi, \xi_1)\phi^{\Delta^m}(\xi), \ \textit{for} \ \ \textit{all}\ \xi \in (\xi_1, \infty)_{\mathbb{T}}.$$

The next lemma is extracted from [12, Lemma 2.2].

Lemma 3. *Let conditions* (H1)–(H4) *hold and let* $\phi(t)$ *be a positive solution of equation* (1). *If*

$$\text{(H5)} \qquad \int_{\xi_0}^{\infty} \left[\int_{v}^{\infty} \left(b^{-1}(s) \int_{s}^{\infty} q(u)\Delta u \right)^{\frac{1}{\alpha}} \Delta s \right] \Delta v = \infty,$$

then there exists $\xi_1 \in [\xi_0, \infty)_{\mathbb{T}}$ *such that for* $t \in [\xi_1, \infty)_{\mathbb{T}}$ *and* $i = 0, 1, \ldots, n-1$, *we have*

$$\phi^{\Delta^i}(\xi) > 0.$$

3. Main Results

We are now ready to establish our first oscillation result in this paper for the equation (1) via a comparison with an n-th order delay differential inequality.

Theorem 1. *Let conditions* (H1)–(H5) *hold. If the n-th order delay differential inequality*

$$\begin{aligned}(4)\quad &\left(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\right)^{\Delta} \\ &\quad + \frac{1}{\alpha} b^{\frac{1-\alpha}{\alpha}}(\tau(\xi))(h_{n-1}(\tau(\xi),\xi_1))^{\alpha-1} q(\xi)\phi^{\gamma+1-\alpha}(\tau(\xi)) \leq 0,\ \xi \geq \xi_1,\end{aligned}$$

has no eventually positive solutions for any large $\xi_1 \geq \xi_0$, then the equation (1) *is oscillatory.*

Proof. Let $\phi(\xi)$ be a nonoscillatory solution of equation (1) with $\phi(t) > 0$ and $\phi(\tau(\xi)) > 0$ for $\xi \geq \xi_1$ for some $\xi_1 \geq \xi_0$. In this proof and the others in this paper, we will only give the details for the case where $\phi(\xi)$ is eventually positive since the proof in the eventually negative case is similar. From (1), it follows that

$$(5)\qquad \left(b(\xi)\left(\phi^{\Delta^{n-1}}(\xi)\right)^{\alpha}\right)^{\Delta} = -q(\xi)\phi^{\gamma}(\tau(\xi)) < 0,$$

so from the fact that $b^{\Delta}(\xi) \geq 0$, it is not difficult to see that there exists $\xi_2 \geq \xi_1$ such that

$$\phi^{\Delta}(\xi) > 0,\ \phi^{\Delta^{n-1}}(\xi) > 0,\ \ \left(b(\xi)\left(\phi^{\Delta^{n-1}}(\xi)\right)^{\alpha}\right)^{\Delta} < 0$$

and

$$\phi^{\Delta^{n}}(\xi) < 0,$$

for $\xi \geq \xi_2$.

Notice that if (H5) holds, Lemma 3 implies $m = n-1$ in Kiguradze's lemma, so that from Lemma 2, we have

$$\phi^{\Delta}(\xi) \geq h_{n-2}(\xi,\xi_1)\phi^{\Delta^{n-1}}(\xi),\ \text{for all}\ \xi \in (\xi_1,\infty)_{\mathbb{T}}.$$

Integrating, this implies

$$(6)\qquad \phi(\xi) \geq h_{n-1}(\xi,\xi_1)\phi^{\Delta^{n-1}}(\xi),\ \text{for all}\ \xi \in (\xi_1,\infty)_{\mathbb{T}}.$$

Now inequality (5) can be written as

$$\left(\left(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\right)^{\alpha}\right)^{\Delta} + q(\xi)\phi^{\gamma}(\tau(\xi)) \leq 0.$$

Taking the Δ-derivative and applying [10, Lemma 2.4], we see that

$$\begin{aligned}\left((b(\xi)\left(\phi^{\Delta^{n-1}}(\xi)\right)^{\alpha}\right)^{\Delta} &= \left(\left(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\right)^{\alpha}\right)^{\Delta} \\ &\geq \alpha\left(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\right)^{\alpha-1}\left(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\right)^{\Delta}.\end{aligned}$$

Hence,

$$(7)\qquad \left(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\right)^{\Delta} + \frac{1}{\alpha}\left(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\right)^{1-\alpha} q(\xi)\phi^{\gamma}(\tau(\xi)) \leq 0,$$

for $\xi \geq \xi_2$. Since $\phi(t) > 0$ and $\phi^{\Delta}(\xi) > 0$, from (6), there is $\xi_3 \geq \xi_2$ with

$$\phi(\xi) \geq h_{n-1}(\xi, \xi_1)\phi^{\Delta^{n-1}}(\xi),$$

for $\xi \geq \xi_3$. From this we see that

$$\phi(\tau(\xi)) \geq h_{n-1}(\tau(\xi), \xi_1)\phi^{\Delta^{n-1}}(\tau(\xi)), \tag{8}$$

for $\xi \geq \xi_4$, where $\tau(\xi) \geq \xi_3$ for $\xi \geq \xi_4$ for some $\xi_4 \geq \xi_3$.

Using the fact that $b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)$ is nonincreasing, we obtain

$$b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi) \leq b^{\frac{1}{\alpha}}(\tau(\xi))\phi^{\Delta^{n-1}}(\tau(\xi)) \tag{9}$$

and using (8) in (9), we obtain

$$b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi) \leq b^{\frac{1}{\alpha}}(\tau(\xi))\phi^{\Delta^{n-1}}(\tau(\xi)) \leq b^{\frac{1}{\alpha}}(\tau(\xi))(h_{n-1}(\tau(\xi), \xi_1))^{-1}\phi(\tau(\xi)).$$

Substituting this inequality into (7) and using the fact that $\alpha > 1$, we see that

$$\Big(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\Big)^{\Delta} + \frac{1}{\alpha}\Big(b^{\frac{1}{\alpha}}(\tau(\xi))(h_{n-1}(\tau(\xi), \xi_1))^{-1}\phi(\tau(\xi))\Big)^{1-\alpha} q(\xi)\phi^{\gamma}(\tau(\xi)) \leq 0,$$

or

$$\Big(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\Big)^{\Delta} + \frac{1}{\alpha}\Big(b^{\frac{1}{\alpha}}(\tau(\xi))(h_{n-1}(\tau(\xi), \xi_1))^{-1}\Big)^{1-\alpha} q(\xi)\phi^{\gamma+1-\alpha}(\tau(\xi)) \leq 0.$$

We then have that $\phi(\xi)$ is a positive solution of (4), which is a contradiction. This completes the proof of the theorem. □

The following result is an immediate consequence of the theorem.

Corollary 1. *Let $\alpha = \gamma$ and conditions* (H1)–(H5) *hold. If the n-th order linear delay dynamic inequality*

$$\Big(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\Big)^{\Delta} + \frac{1}{\alpha}b^{\frac{1-\alpha}{\alpha}}(\tau(\xi))(h_{n-1}(\tau(\xi), \xi_1))^{\alpha-1}q(\xi)\phi(\tau(\xi)) \leq 0,\ \xi \geq \xi_1,$$

has no eventually positive solutions for all $\xi_1 \geq \xi_0$, then the equation (1) *is oscillatory.*

By specializing the choice of time scales, we obtain the following two corollaries. The first one is for the time scale being the real numbers.

Corollary 2. *Let $\mathbb{T} = \mathbb{R}$, $\alpha = \gamma$, and conditions* (H1)–(H5) *hold. If the n-th order linear delay differential inequality*

$$\Big(b^{\frac{1}{\alpha}}(t)\phi^{(n-1)}(\xi)\Big)' + \frac{1}{\alpha}b^{\frac{1-\alpha}{\alpha}}(\tau(\xi))\Big(\frac{(\tau(\xi) - \xi_1)^{n-1}}{(n-1)!}\Big)^{\alpha-1} q(\xi)\phi(\tau(\xi)) \leq 0, \quad \xi \geq \xi_1, \tag{10}$$

has no eventually positive solutions for any large $\xi_1 \geq \xi_0$*, then the differential equation*

$$\Big(b(\xi)\Big(\phi^{(n-1)}(\xi)\Big)^{\alpha}\Big)' + q(\xi)\phi^{\gamma}(\tau(\xi)) = 0,\ \xi \in \mathbb{R},$$

is oscillatory.

On the other hand, if we have $\mathbb{T} = \mathbb{Z}$ so that we are considering difference equations, we then have the following corollary.

Corollary 3. *Let* $\mathbb{T} = \mathbb{Z}$*,* $\alpha = \gamma$*, and conditions* (H1)–(H5) *hold. If the n-th order linear delay difference inequality*

$$\begin{aligned}(11)\quad & \Delta\Big(b^{\frac{1}{\alpha}}(\xi)\phi^{n-1}(\xi)\Big) \\ & \qquad + \frac{1}{\alpha}b^{\frac{1-\alpha}{\alpha}}(\tau(\xi))\Big(\frac{(\tau(\xi)-\xi_1)^{\underline{n-1}}}{(n-1)!}\Big)^{\alpha-1} q(\xi)\phi(\tau(\xi)) \leq 0, \quad \xi \geq \xi_1,\end{aligned}$$

has no eventually positive solutions for any large $\xi_1 \geq \xi_0$*, then the difference equation*

$$\Delta\Big(b(\xi)\Big(\Delta^{n-1}\phi(\xi)\Big)^{\alpha}\Big) + q(\xi)\phi^{\gamma}(\tau(\xi)) = 0,\ \xi \in \mathbb{N},$$

is oscillatory.

Results that guarantee that inequalities (10) or (11) have no positive solutions can be found in a number of places, for example, the monographs of Agarwal *et al.* [2, 4] Györi and Ladas [27], or the papers [28, 30].

Our second oscillation theorem for equation (1) makes a comparison to a first order delay dynamic equation.

Theorem 2. *Let conditions* (H1)–(H5) *hold. If the first order delay dynamic inequality*

$$(12)\qquad w^{\Delta}(\xi) + \frac{1}{\alpha}\frac{q(\xi)(h_{n-1}(\tau(\xi),\xi_1))^{\gamma}}{b^{\frac{\gamma}{\alpha}}(\tau(\xi))}w^{\gamma+1-\alpha}(\tau(\xi)) \leq 0, \quad \xi \geq \xi_1,$$

has no eventually positive solutions for all large $\xi_1 \geq \xi_0$*, then equation* (1) *is oscillatory.*

Proof. Let $\phi(\xi)$ be a nonoscillatory solution of (1), say $\phi(\xi) > 0$ and $\phi(\tau(\xi)) > 0$ for $\xi \geq \xi_1$ for some $\xi_1 \geq \xi_0$. Following the proof of Theorem 1, we again obtain inequalities (7)–(9). Using (8) in (7), we obtain

$$\begin{aligned}&\Big(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\Big)^{\Delta} \\ &\qquad + \frac{1}{\alpha}\Big(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\Big)^{1-\alpha} q(\xi)[h_{n-1}(\tau(\xi),\xi_1)\phi^{\Delta^{n-1}}(\tau(\xi))]^{\gamma} \leq 0\end{aligned}$$

or

$$\left(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\right)^{\Delta}+\frac{1}{\alpha}\left(b^{\frac{1}{\alpha}}(\tau(\xi))\phi^{\Delta^{n-1}}(\xi)\right)^{1-\alpha}\frac{q(\xi)(h_{n-1}(\tau(\xi),\xi_1))^{\gamma}}{b^{\frac{\gamma}{\alpha}}(\tau(\xi))}$$
$$\times\left(b^{\frac{1}{\alpha}}(\tau(\xi))\phi^{\Delta^{n-1}}(\tau(\xi))\right)^{\gamma}\leq 0,$$

and so,

$$\left(b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)\right)^{\Delta}$$
$$+\frac{1}{\alpha}\frac{q(\xi)(h_{n-1}(\tau(\xi),\xi_1))^{\gamma}}{b^{\frac{\gamma}{\alpha}}(\tau(\xi))}\left(b^{\frac{1}{\alpha}}(\tau(\xi))\phi^{\Delta^{n-1}}(\tau(\xi))\right)^{\gamma+1-\alpha}\leq 0.$$

Letting $w(\xi)=b^{\frac{1}{\alpha}}(\xi)\phi^{\Delta^{n-1}}(\xi)$, this becomes

$$w^{\Delta}(\xi)+\frac{1}{\alpha}\frac{q(\xi)(h_{n-1}(\tau(\xi),\xi_1))^{\gamma}}{b^{\frac{\gamma}{\alpha}}(\tau(\xi))}w^{\gamma+1-\alpha}(\tau(\xi))\leq 0.$$

Then w is a positive solution of (12), which is a contradiction and completes the proof of the theorem. □

The following result is immediate.

Corollary 4. *Let conditions* (H1)–(H5) *hold. If*

$$\limsup_{\xi\to\infty}\int_{\tau(\xi)}^{\xi}\frac{q(s)(h_{n-1}(\tau(s),\xi_1))^{\gamma}}{b^{\frac{\gamma}{\alpha}}(\tau(s))}\Delta s=\infty,$$

then the equation (1) *is oscillatory.*

References

[1] R. P. Agarwal, M. Bohner, *Basic calculus on time scales and some of its applications*, Results in Mathematics, 35 (1999), 3–22.

[2] R. P. Agarwal, M. Bohner, S. R. Grace, D. O'Regan, *Discrete Oscillation Theory*, Hindawi, New York, 2005.

[3] R. P. Agarwal, S. R. Grace, *The oscillation of higher-order differential equations with deviating arguments*, Computers & Mathematics with Applications, 38 (1999), 185–199.

[4] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation Theory for Difference and Functional Differential Equations*, Kluwer, Dordrecht, 2000.

[5] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation criteria for certain nth order differential equations with deviating arguments*, Journal of Mathematical Analysis and Applications, 262 (2001), 601–622.

[6] R. P. Agarwal, S. R. Grace, D. O'Regan, *The oscillation of certain higher-order functional differential equations*, Mathematical and Computer Modelling, 37 (2003), 705–728.

[7] R. P. Agarwal, S. R. Grace, D. O'Regan, *Oscillation of certain fourth order functional differential equations*, Ukrainian Mathematical Journal, 59 (2007), 315–342.

[8] M. Bohner, A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[9] M. Bohner, A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[10] D.-X. Chen, *Oscillation of second-order Emden-Fowler neutral delay dynamic equations on time scales*, Mathematical and Computer Modelling, 51 (2010), 1221–1229.

[11] D.-X. Chen, C.Y. Nie, *Oscillation of even order nonlinear functional dynamic equations on time-scales*, IAENG International Journal of Applied Mathematics, 46 (4) (2016), 535–544.

[12] D.-X. Chen, P.-X. Qu, *Oscillation of even order advanced type dynamic equations with mixed nonlinearities on time scales*, Journal of Applied Mathematics and Computing, 44 (2014), 357–377.

[13] L. Erbe, B. Karpuz, A. C. Peterson, *Kamenev-type oscillation criteria for higher-order neutral delay dynamic equations*, International Journal of Difference Equations, 6 (2011), 1–16.

[14] L. Erbe, R. Mert, A. Peterson, A. Zafer, *Oscillation of even order nonlinear delay dynamic equations on time scales*, Czechoslovak Mathematical Journal, 63 (2013), 265–279.

[15] S. R. Grace, *On the oscillation of higher order dynamic equations*, Journal of Advanced Research 4 (2013), 201–204.

[16] S. R. Grace, *On the oscillation of n-order dynamic equations on time-scales*, Mediterranean Journal of Mathematics, 10 (2013), 147–156.

[17] S. R. Grace, R. P. Agarwal, M. Bohner, D. O'Regan, *Oscillation of second order strongly superlinear and strongly sublinear dynamic equations*, Communications in Nonlinear Sciences and Numerical Simulation, 14 (2009), 3463–3471.

[18] S. R. Grace, R. P. Agarwal, B. Kaymakçalan, *Oscillation criteria for even order dynamic equations on time scales*, International Journal of Pure and Applied Mathematics, 72 (2011), 591–597.

[19] S. R. Grace, R. P. Agarwal, B. Kaymakçalan, and W. Sae-jie, *On the oscillation of certain second order nonlinear delay dynamic equations*, Mathematical and Computer Modelling, 50 (2009), 273–286.

[20] S. R. Grace, R. P. Agarwal, A. Zafer, *Oscillation of higher order nonlinear dynamic equations on time scales*, Advances in Difference Equations, 2012 (2012), No. 67.

[21] S. R. Grace, M. Bohner, R. P. Agarwal, *On the oscillation of second-order half-linear dynamic equations*, Journal of Difference Equations and Applications, 15 (2009), 451–460.

[22] S. R. Grace, J. R. Graef, M. A. El-Beltagy, *On the oscillation of third order neutral delay dynamic equations on time scales*, Computers & Mathematics with Applications, 63 (2012), 775–782.

[23] S. R. Grace, T. Hassan, *Oscillation criteria for higher order nonlinear dynamic equations*, Mathematische Nachrichten 287 (2014), 1659–1673.

[24] S. R. Grace, I. Jadlovskà, *Oscillatory behavior of odd-order nonlinear differential equations with a nonpositive neutral term*, Dynamic Systems and Applications, 27 (2018), 125–136.

[25] J. R. Graef, S. R. Grace, E. Tunç, *Oscillatory behavior of even-order nonlinear differential equations with a sublinear neutral term*, Opuscula Mathematica, 39 (2019), 39–47.

[26] J. R. Graef, S. R. Grace, E. Tunç, *Oscillation criteria for even-order differential equations with unbounded neutral coefficients and distributed deviating arguments*, Functional Differential Equations, 25 (2018), 143–153.

[27] I. Györi, G. Ladas, *Oscillation Theory of Delay Differential Equations with Applications*, Oxford University Press, London, 1991.

[28] T. Kusano, M. Naito, *Comparison theorems for functional differential equations with deviating arguments*, Journal of the Mathematical Society of Japan, 3 (1981), 509–532.

[29] T. Li and Y. V. Rogovchenko, *Oscillation criteria for even-order neutral differential equations*, Applied Mathematics Letters, 61 (2016), 35–41.

[30] E. Thandapani, K. Ravi, J. R. Graef, *Oscillation and comparison theorem for halflinear second order difference equations*, Computers & Mathematics with Applications, 42 (2001), 953–960.

[31] A. Zafer, *Oscillation criteria for even order neutral differential equations*, Applied Mathematics Letters, 11 (1998), 21–25.

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